Pergamon

Int. J. Solids Structures Vol. 31, No. 22, pp. 3099–3111, 1994
Copyright © 1994 Elsevier Science Ltd
Printed in Great Britain. All rights reserved
0020–7683/94 $7.00 + .00

0020–7683(94)E0063–2

MICROMECHANICAL ESTIMATES OF THE OVERALL THERMOELECTROELASTIC MODULI OF MULTIPHASE FIBROUS COMPOSITES

TUNGYANG CHEN
Department of Civil Engineering, National Cheng Kung University, Tainan, Taiwan 70101

(*Received* 25 *October* 1993; *in revised form* 6 *April* 1994)

Abstract—Simple algebraic formulae are derived for estimates of the overall thermoelectroelastic moduli of multiphase fibrous composites with the self-consistent and Mori–Tanaka methods. The results are formulated with the help of an overall constraint tensor, which can be obtained analytically by solving a problem for a cylindrical cavity in a piezoelectric medium. It is shown that, although the two methods are different in nature, their estimates of the effective elastic, piezoelectric and dielectric constants, as well as the thermal stress tensor and pyroelectric coefficients of the composite, have a similar structure. In addition, when the phases have equal transverse rigidities in shear, the overall electroelastic moduli predicted by both methods reproduce the *exact* solutions for composites with arbitrary transverse geometry.

1. INTRODUCTION

Piezoelectric composites are an important branch of modern engineering materials, with wide applications in actuators and sensors in "smart" materials and structures. Among various types of piezocomposites, the rod composite, consisting of long, thin rods of piezoelectric ceramic in a matrix, was identified as most promising for ultrasonics (Gururaja *et al.*, 1981). For example, composite sensors containing piezoelectric ceramics rods in a polymer-based matrix are widely used in hydrophones and medical ultrasonic transducers with enhanced mechanical performance, electromechanical coupling and acoustic impedance over the original piezoelectric materials. An extensive review of the technological advantages offered by piezocomposites is given by Smith (1989).

Estimates of overall moduli of piezocomposites in terms of phase moduli, volume fractions and phase geometry are an important topic in the designing and manufacturing process. The objective of this paper is to evaluate the overall thermoelectroelastic moduli of fibrous reinforced composites by the self-consistent and Mori–Tanaka approximations. Specifically, we consider multiphase composite systems reinforced by aligned circular fibers, in which the constituents could be transversely isotropic. The derivation is based on the concept of an "overall constraint" tensor, originally devised by Hill (1965a) for an elastic inclusion. We demonstrate that its analogue in a piezoelectric matrix can be resolved analytically for a cylindrical cavity. Simple algebraic formulae are given for the effective electroelastic moduli and thermal stress tensor as well as pyroelectric coefficients. It is found that, although the two methods are different in nature, their estimates of the thermoelectroelastic moduli have a similar structure. In addition, when the phases have equal transverse shear rigidities, the overall moduli estimated by both methods are identical with the *exact* solutions established by Chen (1993a) for composites with arbitrary transverse geometry.

Recent developments of micromechanical modelling of piezoelectric composites include the work of Grekov *et al.* (1989) who assumed that each fiber and its surrounding cylindrical matrix are located in a medium having effective properties. Wang (1992) examined the piezoelectric inhomogeneity problem and utilized the solutions in calculating the effective constants of fibrous composites without considering phase interactions. Getman and Mol'kov (1992) used an averaging method to study the piezoelectric fibrous composites with periodic structures. Dunn and Taya (1993a) evaluated the effective properties of two-phase composites using dilute, self-consistent, Mori–Tanaka and different micromechanical models. In addition, results of exact nature regarding piezoelectric composites include the works of Schulgasser (1992), Benveniste (1993c), Chen (1993b) and Dunn (1994).

A related problem arises in evaluation of the effective thermal stress tensor and pyroelectric coefficients. Benveniste (1993a) and Dunn (1993a) showed that these tensors are related to the corresponding isothermal electroelastic moduli in two-phase media. Dunn (1993b) further evaluated the numerical results by various micromechanics theories. For multiphase media, Benveniste (1993b) showed that the effective thermal stress and pyroelectric coefficients follow from a knowledge of the influence functions related to an electromechanical loading of the composite aggregate.

The plan of the work is as follows. First, we define the composite system and the phase properties. A summary of most of the present results is given in Section 3. This is followed by an outline of the methods and their formulations for the considered system. Auxiliary boundary value problems for the piezoelectric overall constraint tensor are examined in Section 6.

2. SPECIFICATION OF THE COMPOSITE

We consider a composite medium which consists of a certain number of perfectly bonded homogeneous piezoelectric phases. The inclusions are of cylindrical shape with circular section and each of the phases is transversely isotropic about the "fiber" direction x_3 of a Cartesian coordinate system. In the transverse x_1x_2-plane, the distributions of the phases can be arbitrary, providing that all such transverse sections are identical and the composite can be regarded as statistically homogeneous. Overall transverse isotropy about the x_3-axis is assumed for the composite medium. The constitutive relation for linear piezoelectric materials can be written in the form (Tiersten, 1969)

$$\begin{aligned}\sigma_{ij} &= L_{ijkl}\varepsilon_{kl}-e_{kij}E_k-\lambda_{ij}\theta\\ D_i &= e_{ikl}\varepsilon_{kl}+\kappa_{ik}E_k-q_i\theta,\end{aligned}\tag{1}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\varepsilon}$ the strain tensor, $\mathbf{D}$ the electric displacement vector and $\mathbf{E}$ the electric field. $\mathbf{L}$ are the elastic moduli measured in a constant electric field, $\boldsymbol{\kappa}$ are the dielectric constants measured at constant strain, $\mathbf{e}$ are the piezoelectric constants, $\boldsymbol{\lambda}$ denotes the thermal stress tensor, $\mathbf{q}$ is the vector of pyroelectric coefficients and θ is an increase in the temperature from some reference temperature. The material constants $\mathbf{L}$, $\mathbf{e}$, $\boldsymbol{\kappa}$ are, respectively, fourth-rank, third-rank and second-rank tensors, which satisfy the symmetry relations

$$L_{ijkl} = L_{jikl} = L_{ijlk} = L_{klij}, \quad e_{ikl} = e_{ilk}, \quad \kappa_{ij} = \kappa_{ji}. \tag{2}$$

If $\mathbf{u}$ is the displacement vector and ϕ is the electric potential, the strain tensor and electric field are given by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}), \quad E_i = -\phi_{,i}, \tag{3}$$

where the comma followed by an index indicates the derivative with respect to the corresponding space coordinate. The stress and electric displacement should satisfy the divergence equations $\nabla\cdot\boldsymbol{\sigma} = 0$, $\nabla\cdot\mathbf{D} = 0$. Here the body force and the extrinsic charge are neglected. It is often convenient to write eqn (1) in a matrix notation according to the scheme $11 \equiv 1$, $22 \equiv 2$, $33 \equiv 3$, 23 or $32 \equiv 4$, 31 or $13 \equiv 5$, 12 or $21 \equiv 6$. Accordingly, eqn (1) can be written in the form

$$\begin{bmatrix} \boldsymbol{\sigma} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{L} & \mathbf{e}^{\mathrm{T}} \\ \mathbf{e} & -\boldsymbol{\kappa} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ -\mathbf{E} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{q} \end{bmatrix} \theta, \tag{4}$$

or symbolically as $\hat{\boldsymbol{\sigma}} = \hat{\mathbf{L}}\hat{\boldsymbol{\varepsilon}} - \hat{\boldsymbol{\lambda}}\theta$, where

$$\begin{aligned} \sigma_m &= \sigma_{ij} \quad \text{for} \quad m = 1\text{–}6, \quad i,j = 1,2,3 \\ \varepsilon_m &= \varepsilon_{ij} \quad \text{for} \quad i = j, \quad m = 1,2,3; \quad \varepsilon_m = 2\varepsilon_{ij} \quad \text{for} \quad i \neq j, \quad m = 4,5,6 \\ L_{mn} &= L_{ijkl} \quad \text{for} \quad i,j,k,l = 1,2,3, \quad m,n = 1\text{–}6 \\ e_{ikl} &= e_{in} \quad \text{for} \quad i,k,l = 1,2,3, \quad n = 1\text{–}6 \\ \lambda_m &= \lambda_{ij} \quad \text{for} \quad m = 1\text{–}6, \quad i,j = 1,2,3 \end{aligned} \tag{5}$$

and

$$\hat{\boldsymbol{\sigma}} = \begin{bmatrix} \boldsymbol{\sigma} \\ \mathbf{D} \end{bmatrix}, \quad \hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} \boldsymbol{\varepsilon} \\ -\mathbf{E} \end{bmatrix}, \quad \hat{\mathbf{L}} = \begin{bmatrix} \mathbf{L} & \mathbf{e}^{\mathrm{T}} \\ \mathbf{e} & -\boldsymbol{\kappa} \end{bmatrix}, \quad \hat{\boldsymbol{\lambda}} = \begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{q} \end{bmatrix}. \tag{6}$$

For the considered phase properties which are transversely isotropic about x_3, the constitutive relations can be written in the following form:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} k+m & k-m & l & 0 & 0 & 0 \\ k-m & k+m & l & 0 & 0 & 0 \\ l & l & n & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} - \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \theta$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} + \begin{bmatrix} \kappa_{11} & 0 & 0 \\ 0 & \kappa_{11} & 0 \\ 0 & 0 & \kappa_{33} \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} - \begin{bmatrix} 0 \\ 0 \\ q_3 \end{bmatrix} \theta, \tag{7}$$

where k, l, n, m and p are Hill's (1964) elastic moduli under a constant electric field, λ_1 and λ_3 are the linear thermal stress coefficients in the transverse plane and in the longitudinal direction, respectively, and q_3 is the pyroelectric coefficient. The constitutive forms [eqn (7)] correspond to those of a crystal belonging to the class 6 mm of the hexagonal system, which represents a wide class of technologically important materials in constructing composite piezoelectrics, e.g. PZT (lead zirconate titanate). A particular form, which will be useful in what follows, relates the axisymmetric stress and strain invariants of the transversely isotropic medium:

$$\begin{aligned} s &= ke + l\varepsilon - e_{31}E - \lambda_1\theta \\ \sigma &= le + n\varepsilon - e_{33}E - \lambda_3\theta \\ D &= e_{31}e + e_{33}\varepsilon + \kappa_{33}E - q_3\theta \\ (\sigma_{11} - \sigma_{22}) &= 2m(\varepsilon_{11} - \varepsilon_{22}), \quad \sigma_{12} = 2m\varepsilon_{12}, \end{aligned} \tag{8}$$

in which the strain and stress invariants are defined as

$$e = \varepsilon_{11}+\varepsilon_{22}, \quad \varepsilon = \varepsilon_{33}, \quad E = E_3$$
$$s = (\sigma_{11}+\sigma_{22})/2, \quad \sigma = \sigma_{33}, \quad D = D_3. \tag{9}$$

The remaining constitutive relations involving the three constants p, e_{15} and κ_{11} are linked by

$$\begin{bmatrix}\sigma_4\\ D_2\end{bmatrix} = \begin{bmatrix} p & e_{15}\\ e_{15} & -\kappa_{11}\end{bmatrix}\begin{bmatrix}\varepsilon_4\\ -E_2\end{bmatrix}, \quad \begin{bmatrix}\sigma_5\\ D_1\end{bmatrix} = \begin{bmatrix} p & e_{15}\\ e_{15} & -\kappa_{11}\end{bmatrix}\begin{bmatrix}\varepsilon_5\\ -E_1\end{bmatrix}. \tag{10}$$

In the following analysis we shall conveniently write the (2×2) stiffness matrix as $\tilde{\mathbf{L}}$.

3. OUTLINE OF PRESENT RESULTS

We first summarize the main results for the effective thermoelectroelastic properties of the considered composite system. Derivation of the results will appear in Section 5.

3.1. *Self-consistent estimates*

We consider a system reinforced by aligned, transversely isotropic fibers ($r = 2, 3 \ldots N$) in a transversely isotropic matrix ($r = 1$). Many different fiber materials may be admitted at the same time. The overall thermoelectroelastic moduli of such a system predicted by the self-consistent scheme are

$$k = \frac{\sum_{r=1}^{N} \frac{c_r k_r}{k_r+m}}{\sum_{r=1}^{N} \frac{c_r}{k_r+m}}, \quad l = \frac{\sum_{r=1}^{N} \frac{c_r l_r}{k_r+m}}{\sum_{r=1}^{N} \frac{c_r}{k_r+m}}, \quad e_{31} = \frac{\sum_{r=1}^{N} \frac{c_r e_{31}^r}{k_r+m}}{\sum_{r=1}^{N} \frac{c_r}{k_r+m}} \tag{11}$$

$$m = \frac{\sum_{r=1}^{N} \frac{c_r m_r}{m_r+\gamma}}{\sum_{r=1}^{N} \frac{c_r}{m_r+\gamma}}, \quad \gamma = \left(\frac{1}{m}+\frac{2}{k}\right)^{-1} \tag{12}$$

$$n = \sum_{r=1}^{N} c_r n_r - \sum_{r=1}^{N} \frac{c_r l_r^2}{k_r+m} + \frac{\left[\sum_{r=1}^{N} \frac{c_r l_r}{k_r+m}\right]^2}{\sum_{r=1}^{N} \frac{c_r}{k_r+m}} \tag{13}$$

$$e_{33} = \sum_{r=1}^{N} c_r e_{33}^r - \sum_{r=1}^{N} \frac{c_r l_r e_{31}^r}{k_r+m} + \frac{\sum_{r=1}^{N} \frac{c_r e_{31}^r}{k_r+m} \sum_{r=1}^{N} \frac{c_r l_r}{k_r+m}}{\sum_{r=1}^{N} \frac{c_r}{k_r+m}} \tag{14}$$

$$\kappa_{33} = \sum_{r=1}^{N} c_r \kappa_{33}^r + \sum_{r=1}^{N} \frac{c_r (e_{31}^r)^2}{k_r+m} - \frac{\left[\sum_{r=1}^{N} \frac{c_r e_{31}^r}{k_r+m}\right]^2}{\sum_{r=1}^{N} \frac{c_r}{k_r+m}} \tag{15}$$

$$\tilde{\mathbf{L}} = \frac{1}{2}\left[\sum_{r=1}^{N} c_r(\tilde{\mathbf{L}}_r+\tilde{\mathbf{L}})^{-1}\right]^{-1}. \tag{16}$$

Clearly, among the seven effective constants of the composite, k, l, n, e_{31}, e_{33}, κ_{33} and m, the moduli k and m should be resolved first. In particular, the knowledge of m will suffice to determine the other six moduli. Remarkably, when all phases have equal rigidities in shear, the overall shear modulus is just m itself and these six moduli are exactly the same expressions with the exact results (Chen, 1993a) for fiber-strengthened piezoelectric materials with equal shear rigidities in which the transverse geometry could be arbitrary. Further, it is observed that the remaining three constants p, κ_{11} and ε_{15} are independent of eqns (11)–(15).

We now list the corresponding results [eqns (11)–(15)] for two-phase systems of technological interest; the subscripts f and m represent the fiber and matrix, respectively:

$$k = \frac{c_f k_f(k_m+m)+c_m k_m(k_f+m)}{c_f(k_m+m)+c_m(k_f+m)}, \quad l = \frac{c_f l_f(k_m+m)+c_m l_m(k_f+m)}{c_f(k_m+m)+c_m(k_f+m)} \tag{17}$$

$$m = \frac{m_m m_f(k+2m)+km(c_f m_f+c_m m_m)}{km+(k+2m)(c_f m_m+c_m m_f)}, \quad e_{31} = \frac{c_f e^{f}_{31}(k_m+m)+c_m e^{m}_{31}(k_f+m)}{c_f(k_m+m)+c_m(k_f+m)} \tag{18}$$

$$n = c_f n_f + c_m n_m + \frac{l^2}{k+m} - \frac{c_f l_f^2}{k_f+m} - \frac{c_m l_m^2}{k_m+m} \tag{19}$$

$$e_{33} = c_f e^{f}_{33} + c_m e^{m}_{33} + \frac{l e_{31}}{k+m} - \frac{c_f e^{f}_{31} l_f}{k_f+m} - \frac{c_m e^{m}_{31} l_m}{k_m+m} \tag{20}$$

$$\kappa_{33} = c_f \kappa^{f}_{33} + c_m \kappa^{m}_{33} - \frac{e_{31}^2}{k+m} + \frac{c_f(e^{f}_{31})^2}{k_f+m} + \frac{c_m(e^{m}_{31})^2}{k_m+m}. \tag{21}$$

We note that the effective moduli k, l, n and m under a constant electric field take the same forms as those derived by Hill (1965a) and Laws (1974) for purely elastic media. As discussed therein the estimates can be physically unacceptable when the phase properties differ considerably. Thus, it follows that this theory in the context of piezoelectricity remains unreliable under extreme conditions.

We now turn to the effective thermal stress vectors and pyroelectric constants of the composite. These coefficients can be expressed in terms of the concentration factors related to an electromechanical loading of the composite aggregate in which no eigenstresses and polarizations are present. The correspondence is a generalization of Levin's formula in thermoelastic composites (Levin, 1967). We shall show later that the results for effective thermal stress and pyroelectric coefficients of the composite by the self-consistent method are

$$\lambda_1 = \sum_{r=1}^{N} c_r \frac{k+m}{k_r+m}\lambda_1^r, \quad \lambda_3 = \sum_{r=1}^{N} c_r \lambda_3^r + \sum_{r=1}^{N} c_r \frac{(l-l_r)}{k_r+m}\lambda_1^r$$

$$q_3 = \sum_{r=1}^{N} c_r q_3^r + \sum_{r=1}^{N} c_r \frac{(e^r_{31}-e_{31})}{k_r+m}\lambda_1^r. \tag{22}$$

It can be verified that when piezoelectric coupling is absent, the expressions reduce to those of Laws (1974, p. 86) for thermoelastic composite media.

3.2. *The Mori–Tanaka method*

As above, we denote the matrix as $r = 1$, and the fibers as $r = 2, 3, \ldots N$. Transverse isotropy is assumed in all phases, together with alignment of the phase symmetry axes with x_3. The effective physical constants of such composite predicted by the Mori–Tanaka method are

$$k=\frac{\sum_{r=1}^{N}\frac{c_r k_r}{k_r+m_1}}{\sum_{r=1}^{N}\frac{c_r}{k_r+m_1}},\quad l=\frac{\sum_{r=1}^{N}\frac{c_r l_r}{k_r+m_1}}{\sum_{r=1}^{N}\frac{c_r}{k_r+m_1}},\quad e_{31}=\frac{\sum_{r=1}^{N}\frac{c_r e_{31}^r}{k_r+m_1}}{\sum_{r=1}^{N}\frac{c_r}{k_r+m_1}} \tag{23}$$

$$m=\frac{\sum_{r=1}^{N}\frac{c_r m_r}{m_r+\gamma_1}}{\sum_{r=1}^{N}\frac{c_r}{m_r+\gamma_1}},\quad \gamma_1=\left(\frac{1}{m_1}+\frac{2}{k_1}\right)^{-1} \tag{24}$$

$$n=\sum_{r=1}^{N}c_r n_r-\sum_{r=1}^{N}\frac{c_r l_r^2}{k_r+m_1}+\frac{\left[\sum_{r=1}^{N}\frac{c_r l_r}{k_r+m_1}\right]^2}{\sum_{r=1}^{N}\frac{c_r}{k_r+m_1}} \tag{25}$$

$$e_{33}=\sum_{r=1}^{N}c_r e_{33}^r-\sum_{r=1}^{N}\frac{c_r l_r e_{31}^r}{k_r+m_1}+\frac{\sum_{r=1}^{N}\frac{c_r e_{31}^r}{k_r+m_1}\sum_{r=1}^{N}\frac{c_r l_r}{k_r+m_1}}{\sum_{r=1}^{N}\frac{c_r}{k_r+m_1}} \tag{26}$$

$$\kappa_{33}=\sum_{r=1}^{N}c_r\kappa_{33}^r+\sum_{r=1}^{N}\frac{c_r(e_{31}^r)^2}{k_r+m_1}-\frac{\left[\sum_{r=1}^{N}\frac{c_r e_{31}^r}{k_r+m_1}\right]^2}{\sum_{r=1}^{N}\frac{c_r}{k_r+m_1}} \tag{27}$$

$$\tilde{\mathbf{L}}=\left[\sum_{r=1}^{N}c_r(\tilde{\mathbf{L}}_r+\tilde{\mathbf{L}}_1)^{-1}\right]^{-1}-\tilde{\mathbf{L}}_1. \tag{28}$$

A comparison with those estimated by the self-consistent scheme reveals a structural similarity between the two approximate procedures (11)–(16) and (23)–(28). A similar conclusion in the context of elasticity, though in different expressions, was pointed out by Dvorak and Benveniste (1992). The principal distinction between the two methods is that in the self-consistent method, the constant m in the formulae is the effective shear modulus, whereas in the Mori–Tanaka method it is replaced by the matrix shear modulus m_1. Again, it is noteworthy that the moduli (23)–(27) are identical with the known *exact* solutions (Chen, 1993a) for composites with arbitrary transverse geometry, when the phases have equal transverse shear moduli.

For two-phase systems, the results are much simpler. We recorded them here for completeness; again, the subscripts f and m represent the fiber and matrix, respectively:

$$k=\frac{c_f k_f(k_m+m_m)+c_m k_m(k_f+m_m)}{c_f(k_m+m_m)+c_m(k_f+m_m)},\quad l=\frac{c_f l_f(k_m+m_m)+c_m l_m(k_f+m_m)}{c_f(k_m+m_m)+c_m(k_f+m_m)} \tag{29}$$

$$m=\frac{m_m m_f(k_m+2m_m)+k_m m_m(c_f m_f+c_m m_m)}{k_m m_m+(k_m+2m_m)(c_f m_m+c_m m_f)},\quad e_{31}=\frac{c_f e_{31}^f(k_m+m_m)+c_m e_{31}^m(k_f+m_m)}{c_f(k_m+m_m)+c_m(k_f+m_m)} \tag{30}$$

$$n = c_f n_f + c_m n_m + \frac{l^2}{k+m_m} - \frac{c_f l_f^2}{k_f+m_m} - \frac{c_m l_m^2}{k_m+m_m} \tag{31}$$

$$e_{33} = c_f e_{33}^f + c_m e_{33}^m + \frac{l e_{31}}{k+m_m} - \frac{c_f e_{31}^f l_f}{k_f+m_m} - \frac{c_m e_{31}^m l_m}{k_m+m_m} \tag{32}$$

$$\kappa_{33} = c_f \kappa_{33}^f + c_m \kappa_{33}^m - \frac{e_{31}^2}{k+m_m} + \frac{c_f (e_{31}^f)^2}{k_f+m_m} + \frac{c_m (e_{31}^m)^2}{k_m+m_m}. \tag{33}$$

Now, going back to the effective thermal expansion coefficients and pyroelectric constants of the composite, the coefficients estimated by the Mori–Tanaka method are exactly the same expressions as eqn (22) except that the parameter m is replaced by m_1.

4. EFFECTIVE MODULI OF PIEZOELECTRIC COMPOSITES

A representative volume element V of the composite is chosen such that under homogeneous boundary conditions it represents the macroscopic response of the composite. The phase volume fractions c_r satisfy $\Sigma c_r = 1$, $r = 1, 2 \ldots N$. The volume V is subjected to uniform displacement and electric boundary conditions, and a uniform temperature change θ°

$$\mathbf{u}(S) = \varepsilon_{ij}^\circ x_j, \quad \phi(S) = -E_i^\circ x_i, \quad \theta(S) = \theta^\circ, \tag{34}$$

where $\mathbf{u}$ and ϕ denote the applied displacement and electric potential, $\boldsymbol{\varepsilon}^\circ$ and $\mathbf{E}^\circ$ are constant strain and electric field, and $\mathbf{n}$ is the outside normal to S. The overall elastic, piezoelectric and dielectric constants of the composite aggregate are defined by

$$\bar{\boldsymbol{\sigma}} = \mathbf{L}\boldsymbol{\varepsilon}^\circ - \mathbf{e}^T\mathbf{E}^\circ - \boldsymbol{\lambda}\theta, \quad \bar{\mathbf{D}} = \mathbf{e}\boldsymbol{\varepsilon}^\circ + \boldsymbol{\kappa}\mathbf{E}^\circ - \mathbf{q}\theta, \tag{35}$$

where $\bar{\boldsymbol{\sigma}}$ and $\bar{\mathbf{D}}$ denote the volume average stresses and electric displacements in V. Under the boundary condition (34), the local and overall field averages in V are given by

$$\boldsymbol{\varepsilon}^\circ = \sum_{r=1}^{N} c_r \boldsymbol{\varepsilon}_r \quad \mathbf{E}^\circ = \sum_{r=1}^{N} c_r \mathbf{E}_r, \quad \bar{\boldsymbol{\sigma}} = \sum_{r=1}^{N} c_r \boldsymbol{\sigma}_r, \quad \bar{\mathbf{D}} = \sum_{r=1}^{N} c_r \mathbf{D}_r. \tag{36}$$

To evaluate the overall moduli of the composite aggregate, it is often convenient to introduce the phase volume averages $\hat{\boldsymbol{\varepsilon}}_r = \hat{\mathbf{A}}_r \hat{\boldsymbol{\varepsilon}}^\circ$ so that the overall moduli $\hat{\mathbf{L}}$ follows as $\hat{\mathbf{L}} = \Sigma c_r \hat{\mathbf{L}}_r \hat{\mathbf{A}}_r$, where $\hat{\mathbf{A}}_r$ is referred to as concentration factors.

In the evaluation of the concentration factors by the self-consistent scheme, each inclusion is regarded as a solitary inhomogeneity embedded in an infinite effective medium under a remotely applied boundary condition $\hat{\boldsymbol{\varepsilon}}^\circ \mathbf{x}$. For an ellipsoidal inclusion, the local fields in the solitary inhomogeneity are uniform (Wang, 1992; Benveniste, 1992) and the concentration factor $\hat{\mathbf{A}}_r$ can be expressed by

$$\hat{\mathbf{A}}_r = [\hat{\mathbf{I}} + \hat{\mathbf{S}}\hat{\mathbf{L}}^{-1}(\hat{\mathbf{L}}_r - \hat{\mathbf{L}})]^{-1}, \tag{37}$$

where $\hat{\mathbf{I}}$ is a unit diagonal (9×9) matrix and $\hat{\mathbf{S}}$ is the equivalent Eshelby tensor which depends only on the shape of the inclusion and on the properties of the surrounding matrix. An integral form of $\hat{\mathbf{S}}$ for ellipsoidal inclusions was recently formulated by Chen (1993c). In particular, when the inclusion is in the shape of a circular cylinder in a transversely isotropic matrix, Dunn and Taya (1993a) gave explicit formulae for $\hat{\mathbf{S}}$. Its nonvanishing terms are recorded as

$$S_{11} = S_{22} = \frac{6k+4m}{8(k+m)}, \quad S_{12} = S_{21} = \frac{2k-4m}{8(k+m)}, \quad S_{13} = S_{23} = \frac{l}{2(k+m)},$$

$$S_{66} = \frac{k+2m}{2(k+m)}, \quad S_{44} = S_{55} = S_{77} = S_{88} = \frac{1}{2}. \tag{38}$$

The components in eqn (38) are written according to the notation (5) [see also Chen (1993d)]. Alternatively, one may write eqn (37) in terms of an equivalent "overall constraint" tensor $\hat{\mathbf{L}}^*$, the piezoelectric analogue of constraint modulus (Hill, 1965b), which relates the uniform fields in the inclusion r to the uniform fields $\hat{\boldsymbol{\sigma}}^\circ$ and $\hat{\boldsymbol{\varepsilon}}^\circ$ as

$$\hat{\boldsymbol{\sigma}}_r - \hat{\boldsymbol{\sigma}}^\circ = \hat{\mathbf{L}}^*(\hat{\boldsymbol{\varepsilon}}^\circ - \hat{\boldsymbol{\varepsilon}}_r) \quad \text{or equivalently} \quad \hat{\mathbf{A}}_r = (\hat{\mathbf{L}}^* + \hat{\mathbf{L}}_r)^{-1}(\hat{\mathbf{L}}^* + \hat{\mathbf{L}}). \tag{39}$$

Similarly, one may define an overall constraint $\hat{\mathbf{M}}^*$, the inverse of $\hat{\mathbf{L}}^*$, by reversing expression (39). In the following analysis we shall employ this approach to derive the dilute concentration factor. The solutions for $\hat{\mathbf{L}}^*$ or $\hat{\mathbf{M}}^*$ can be obtained by solving boundary value problems for a uniform strained infinite medium containing a cavity. A description of the procedures for the overall constraint compliance for a cylindrical cavity in a transversely isotropic piezoelectric matrix is given in Section 6. The nonvanishing terms of the overall constraint $\mathbf{M}^*$ are obtained as

$$M^*_{11} = M^*_{22} = \frac{1}{2}\left(\frac{1}{m} + \frac{1}{k}\right), \quad M^*_{12} = M^*_{21} = -\frac{1}{2}\frac{1}{k}, \quad M^*_{66} = \frac{1}{m} + \frac{2}{k}$$

$$\begin{bmatrix} M^*_{44} & M^*_{48} \\ M^*_{84} & M^*_{88} \end{bmatrix} = \begin{bmatrix} M^*_{55} & M^*_{57} \\ M^*_{75} & M^*_{77} \end{bmatrix} = \begin{bmatrix} p & e_{15} \\ e_{15} & -\kappa_{11} \end{bmatrix}^{-1}. \tag{40}$$

Now turning to the Mori–Tanaka estimate, the concentration factor is obtained by the assumption that each inclusion is regarded as a solitary inhomogeneity embedded in an infinite matrix material under a remotely applied field equal to the matrix average $\hat{\boldsymbol{\varepsilon}}_1$ or $\hat{\boldsymbol{\sigma}}_1$. Consequently, similar to that of elasticity, the concentration factor can be derived as

$$\hat{\mathbf{A}}_r = \hat{\mathbf{T}}_r\left[\sum_{r=1}^{N} c_r \hat{\mathbf{T}}_r\right]^{-1}, \quad \hat{\mathbf{T}}_r = (\hat{\mathbf{L}}^*_1 + \hat{\mathbf{L}}_r)^{-1}(\hat{\mathbf{L}}^*_1 + \hat{\mathbf{L}}_1). \tag{41}$$

The fundamentals of the two approximate methods are fairly well known in the context of mechanical properties. Their extension to elastoelectric moduli is straightforward and thus complete descriptions are not presented here. For a detailed exposition of the methods, the reader is referred to the works of Dvorak and Benveniste (1992) and Dunn and Taya (1993a).

The effective thermal stress and pyroelectric coefficients are also important physical properties of piezoelectric composites. These coefficients can be derived from the virtual work theorems in piezoelectric media (Benveniste, 1993b) which follow a direct extension of Rosen and Hashin (1970) in deriving the effective thermal expansion coefficients for elastic composite media. In particular, according to our definition of the concentration factor, the results derived by Benveniste [1993b, eqn (13)] can be recast as

$$\hat{\boldsymbol{\lambda}} = \sum_{r=1}^{N} c_r \hat{\mathbf{A}}_r^{\mathrm{T}} \hat{\boldsymbol{\lambda}}_r, \tag{42}$$

which has a similar structure with the effective thermal stress tensor of composites derived by Levin (1967) and Rosen and Hashin (1970).

5. COMPOSITES REINFORCED BY ALIGNED FIBERS

We now proceed to derive the results which were summarized in Section 3. First we derive the results by the self-consistent method. Consider a single fiber in an infinite effective medium subjected to a transverse shear strain 2ε on its outside boundary. In this dilute configuration, it is obvious that the local and overall quantities are related, according to eqns (39) and (40), as $\tau_r - \tau = 2\gamma(\varepsilon - \varepsilon_r)$, where τ_r and ε_r represent the transverse shear stress and strain in the phase r, respectively. From the phase constitutive relation $(8)_4$, one finds that $\tau_r/\tau = [m_r(m+\gamma)]/[m(m_r+\gamma)]$. Consequently, by the average theorem of strain [eqn $(36)_1$], the effective transverse shear modulus m can be derived as

$$\frac{1}{m+\gamma} = \sum_{r=1}^{N} \frac{c_r}{m_r+\gamma}. \tag{43}$$

Next, consider a pure lateral dilatation without longitudinal straining or electric loading, i.e. $e \neq 0$, $\varepsilon = 0$, $E = 0$. The overall behavior is thus reduced to $\bar{s} = ke$, $\bar{\sigma} = le$, $\bar{D} = e_{31}e$. From eqns (39) and (40), the corresponding equation for the constraint modulus is $s_r - s = m(e - e_r)$, which implies that $s_r/s = [k_r(k+m)]/[k(k_r+m)]$, and hence the effective moduli k, l and e_{31} can be found as

$$\frac{1}{k+m} = \sum_{r=1}^{N} \frac{c_r}{k_r+m}, \quad \frac{l}{k+m} = \sum_{r=1}^{N} \frac{c_r l_r}{k_r+m}, \quad \frac{e_{31}}{k+m} = \sum_{r=1}^{N} \frac{c_r e_{31}^r}{k_r+m}. \tag{44}$$

Some algebra will show that eqns (43) and (44) are equivalent to the formulae given in eqns (11) and (12).

To find the effective moduli n and e_{33}, we consider an overall uniaxial straining without lateral contraction, or electric loading, i.e. $\varepsilon \neq 0$, $e = E = 0$. For evaluation of the modulus κ_{33}, we assume an overall loading of $E \neq 0$, $e = \varepsilon = 0$. In addition to eqns (36) and (40), in multiphase systems two additional conditions are needed for evaluation of these constants, namely the quantities of e_r/ε and e_r/E. This can be accomplished either by solving appropriate boundary value problems or by direct expansion of eqn (39). In either approach, the results can be shown as

$$\frac{e_r}{\varepsilon} = \frac{l - l_r}{m+m_k}, \quad \frac{e_r}{E} = \frac{e_{31}^r - e_{31}}{m+k_r}. \tag{45}$$

Accordingly, the effective moduli n, e_{33} and κ_{33} can be derived in the forms given by eqns (13)–(15). The remaining three effective moduli p, e_{15} and κ_{11} are coupled and independent of the other seven constants. Their evaluation procedures follow a standard formulation which involves the substitution of eqn (39) into $\hat{\mathbf{L}} = \Sigma\, c_r \hat{\mathbf{L}}_r \hat{\mathbf{A}}_r$. Some algebra will show that these three effective constants of the composite estimated by the self-consistent method are

$$\hat{\mathbf{L}} = \left[\sum_{r=1}^{N} c_r (\tilde{\mathbf{L}}_r + \tilde{\mathbf{L}}^*)^{-1} \right]^{-1} - \tilde{\mathbf{L}}^*. \tag{46}$$

Since in the present loading conditions $\tilde{\mathbf{L}}^*$ is equal to $\tilde{\mathbf{L}}$, the overall moduli $\tilde{\mathbf{L}}$ is thus reduced to eqn (16).

We now derive the effective moduli by the Mori–Tanaka method. As mentioned previously, the method regards each inclusion as a solitary inhomogeneity embedded in an infinite matrix material under a remotely applied quantity equal to the matrix average $\hat{\varepsilon}_1$. The formulation for the effective properties mainly follows the routes described in the self-consistent scheme except that the overall strain or electric field is replaced by the matrix average $\hat{\varepsilon}_1$, while the quantity $\hat{\varepsilon}_1$ can be obtained from the identity $(36)_1$ and $(36)_2$. The derivation is similar to that of the self-consistent method described above. For a detailed illustration of the procedures, the reader is referred to the work of Chen *et al.* (1992) who derived the effective elastic moduli of multiphase composites by the Mori–Tanaka method.

In view of the results given in eqns (11)–(16) and (23)–(28), it is seen that the effective elastic moduli k, l, n and m are independent of piezoelectric coupling, however, the effective dielectric constants appear to be functions of both elastic and piezoelectric coefficients of the phases. This is not a surprising outcome. In fact, one can recognize this effect directly from the constitutive relations (8). For example, as described above, the effective modulus k is defined to be the ratio of average stress $(\sigma_{11}+\sigma_{22})/2$ in the medium divided by an applied dilatational strain e. The overall electric field E_3 (and hence throughout the whole medium) can, however, be set equal to zero at will. Thus, the constant k is independent of piezoelectric coefficients. Similar arguments apply to l, m and n. On the other hand, an overall loading of $E \neq 0$, $e = \varepsilon = 0$, will induce not only D but also s and σ in the medium. Accordingly, the effective κ_{33} is a function of dielectric, piezoelectric and elastic constants as well. Further, by simple algebra it can be shown that the effective elastic constants k, l, n and m are exactly the same as those for the uncoupled elastic composites (Chen *et al.*, 1992). However, the effective constant p is coupled with κ_{11} and e_{15} as indicated in eqns (16) and (28).

The derivation of the effective thermal stress tensor and pyroelectric coefficients mainly depends on the solutions of $\hat{\mathbf{A}}_r$. In particular, from the previous formulations, the concentration factor can be expressed as

$$\begin{bmatrix} e \\ \varepsilon \\ E \end{bmatrix}_r = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}_r \begin{bmatrix} e^\circ \\ \varepsilon^\circ \\ E^\circ \end{bmatrix}. \tag{47}$$

When the self-consistent approximation is assumed, the nonvanishing components of the concentration factor are

$$A_{11} = \frac{k+m}{m+k_r}, \quad A_{12} = \frac{l-l_r}{m+k_r}, \quad A_{13} = \frac{e_{31}^r - e_{31}}{m+k_r}, \quad A_{22} = A_{33} = 1, \tag{48}$$

whereas the Mori–Tanaka assumption provides

$$A_{11} = \frac{k+m_1}{m_1+k_r}, \quad A_{12} = \frac{l-l_r}{m_1+k_r}, \quad A_{13} = \frac{e_{31}^r - e_{31}}{m_1+k_r}, \quad A_{22} = A_{33} = 1. \tag{49}$$

6. THE OVERALL PIEZOELECTRIC CONSTRAINT TENSOR

The overall constraint compliance quoted in eqn (40) will now be justified. We consider a homogeneous medium containing a cylindrical cavity with its axis parallel to the x_3-axis, where the constitutive equation of the matrix is given in eqn (7). On the cavity surface, the traction vector and the normal component of electric displacement are zero. We first consider a loading case involving the three constants p, e_{15} and κ_{11}. It can be easily verified that, for a cylindrical cavity in a transversely isotropic matrix, the medium admits a two-dimensional elastic and electric field characterized by

$$u_1 = u_2 = E_3 = 0, \quad u_3 = u(r,\theta), \quad \phi = \phi(r,\theta), \tag{50}$$

so that the corresponding nonvanishing stress and electric displacement are

$$\sigma_{rz} = p\frac{\partial u}{\partial r} + e_{15}\frac{\partial \phi}{\partial r}, \quad \sigma_{\phi z} = p\frac{1}{r}\frac{\partial u}{\partial \theta} + e_{15}\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$D_r = e_{15}\frac{\partial u}{\partial r} - \kappa_{11}\frac{\partial \phi}{\partial r}, \quad D_\theta = e_{15}\frac{1}{r}\frac{\partial u}{\partial \theta} - \kappa_{11}\frac{1}{r}\frac{\partial \phi}{\partial \theta}. \tag{51}$$

Since $\sigma_{ij,j} = D_{i,i} = 0$, it can be readily shown that the displacement u and electric potential ϕ satisfy the Laplace equation $\nabla^2 u = \nabla^2 \phi = 0$. By separation of variables, the admissible displacement and electric potential in the medium can be expressed as

$$u = \left(Ar + \frac{B}{r}\right)\sin\theta, \quad \phi = \left(Cr + \frac{D}{r}\right)\sin\theta. \tag{52}$$

The boundary conditions to be satisfied are the vanishing of traction and normal component of electric displacement on the cavity surface, together with the imposed boundary conditions at infinity. These conditions provide the equations for evaluation of the four unknown constants A, B, C and D, which can be easily solved analytically. In derivation of $\tilde{\mathbf{L}}^*$, it is convenient to apply the following two boundary conditions separately:

$$u|_{r\to\infty} = \varepsilon^\circ r\sin\theta, \quad \phi|_{r\to\infty} = 0, \quad u|_{r\to\infty} = 0, \quad \phi|_{r\to\infty} = -E^\circ r\sin\theta. \tag{53}$$

Once the constants are determined, we can evaluate the average strain and electric field in the cavity. Thus, by the definition of eqn (39), the component in eqn $(40)_2$ can then be obtained.

We turn to the constraint compliance for the plane transverse strain. For the present purpose it is enough to consider a medium with a cavity under a homogeneous boundary condition $\sigma_1 = \sigma^0$, $\sigma_2 = 0$, $\sigma_3 = 0$ and $D_i = 0$ applied at infinity. Under this loading condition the field is independent of x_3, which implies that $E_3 = \varepsilon_3 = 0$. Thus the local fields in the matrix reduce to

$$\sigma_1 = (k+m)\varepsilon_1 + (k-m)\varepsilon_2, \quad \sigma_2 = (k-m)\varepsilon_1 + (k+m)\varepsilon_2, \quad \sigma_3 = l(\varepsilon_1 + \varepsilon_2), \quad \sigma_6 = m\varepsilon_6$$

$$D_1 = \kappa_{11}E_1, \quad D_2 = \kappa_{11}E_2, \quad D_3 = e_{31}(\varepsilon_1 + \varepsilon_2). \tag{54}$$

In view of the field eqns (54) and the boundary condition it is obvious that the *elastic* fields associated with the auxiliary boundary value problem are exactly the same as those for purely elastic material. Accordingly, the derivations for the corresponding constraint compliance follow the same procedures illustrated by Hill (1965a). This will provide the components listed in eqn $(40)_1$. It should be noted that the components $M^*_{91}, M^*_{92} \ldots M^*_{99}$ are all identical to zero, since $E'_3 = E_3 = 0$. Likewise, $\varepsilon'_3 = \varepsilon_3 = 0$, so it follows that $M^*_{31} = M^*_{32} = \cdots = M^*_{39} = 0$. We now complete the derivation procedure of the constraint compliance for a circular cylindrical inclusion in a transversely isotropic piezoelectric medium.

An analogous boundary value problem in which the cavity is substituted by a different material could also be resolved in a similar manner. In particular, this will provide the solutions of the dilute concentration factor listed in eqns (48) and (49). We mention that the procedures described could be verified by an alternative approach via eqn (37), which involves the use of the equivalent Eshelby tensor [eqn (38)]. Again, it is stressed that the present formulation using the overall constraint tensor is an alternative approach for the solution of the inclusion problem. For a detailed description of $\hat{\mathbf{L}}^*$, one can refer to the celebrated work of Hill (1965b).

7. CONCLUDING REMARKS

Recently, Nan (1993) developed a general theoretical framework for the determination of the effective properties of piezoelectric composites following the multiple scattering scheme. As an example, he gave explicit first-order estimates (average t-matrix approximation) for the effective properties of binary transversely isotropic fibrous composites, in which the matrix is isotropic (non-piezoelectric). It is surprising to find that their estimates of k, l, m and e_{31} are identical with those predicted by the Mori–Tanaka method [eqns (29) and (30)]. However, the remaining physical constants are different. We notice that, by numerical evaluations, their results [Nan (1993); eqn (4)] do not completely satisfy the exact universal relationships (Schulgasser, 1992) between the effective constants.

We finally remark that both the self-consistent and Mori–Tanaka theories provide estimates which are admissible if the effective constants are diagonally symmetric and self-consistent. Dunn and Taya (1993b) have proven that the Mori–Tanaka theory is on strong footing for two-phase piezocomposites with aligned inclusions. Whether these requirements are fulfilled for various kinds of multicomponent composites remains to be established. However, based on our previous work (Chen *et al.*, 1992) and the analogy between the elastic and piezoelectric composites, we would point out that the effective properties for the considered system can be expressed as

$$\hat{\mathbf{L}} = \left[\sum_{r=1}^{N} c_r (\hat{\mathbf{L}}_r + \hat{\mathbf{L}}_1^*)^{-1} \right]^{-1} - \hat{\mathbf{L}}_1^* \tag{55}$$

for the Mori–Tanaka method. A similar equation can be slightly modified for the self-consistent approximation. We thus conclude that the results still hold for multiphase systems where all inclusions have the same shape and alignment. Also, to further examine the validity of the results, the effective constants need to be bracketed by available variational bounds. In fact, to the author's knowledge, rigorous bounds have yet to be developed for the effective properties of piezocomposites. Nevertheless, in this particular system, the electrical state of the medium does not affect the transverse shear modulus. Hence, the effective shear modulus and its variational bounds are exactly the same as those for the purely elastic media.

Acknowledgement—This work was supported by the National Science Council, Taiwan, under contracts NSC 83–0410–E006–021 and NSC 84–0410–E006–002.

REFERENCES

Benveniste, Y. (1992). The determination of the elastic and electric fields in a piezoelectric inhomogeneity. *J. Appl. Phys.* **72,** 1086–1095.

Benveniste, Y. (1993a). Universal relations in piezoelectric composites with eigenstress and polarization fields. I. Binary media: local fields and effective behavior. *J. Appl. Mech.* **60,** 265–269.

Benveniste, Y. (1993b). Universal relations in piezoelectric composites with eigenstress and polarization fields. II. Multiphase media: effective behavior. *J. Appl. Mech.* **60,** 270–275.

Benveniste, Y. (1993c). Exact results in the micromechanics of fibrous piezoelectric composites exhibiting pyroelectricity. *Proc. R. Soc. London* **441A,** 59–81.

Chen, T. (1993a). Piezoelectric properties of multiphase fibrous composites: some theoretical results. *J. Mech. Phys. Solids* **41,** 1781–1794.

Chen, T. (1993b). An invariant treatment of interfacial discontinuities in piezoelectric media. *Int. J. Engng Sci.* **31,** 1061–1072.

Chen, T. (1993c). Green's functions and the non-uniform transformation problem in a piezoelectric medium. *Mech. Res. Comm.* **20,** 271–278.

Chen, T. (1993d). The rotation of a rigid ellipsoidal inclusion embedded in an anisotropic piezoelectric medium. *Int. J. Solids Structures* **30,** 1983–1995.

Chen, T., Dvorak, G. J. and Benveniste, Y. (1992). Mori–Tanaka estimates of the overall elastic moduli of certain composite materials. *J. Appl. Mech.* **59,** 539–546.

Dunn, M. L. (1993a). Exact relations between the thermoelectroelastic moduli of heterogeneous materials. *Proc. R. Soc. London* **441A,** 549–557.

Dunn, M. L. (1993b). Micromechanics of coupled electroelastic composites: effective thermal expansion and pyroelectric coefficients, *J. Appl. Phys.* **73,** 5131–5140.

Dunn, M. L. (1994). Electroelastic Green's functions for transversely isotropic piezoelectric media and their application to the solution of inclusion and inhomogeneity problems. *Int. J. Engng Sci.* **32,** 119–131.

Dunn, M. L. and Taya, M. (1993a). Micromechanics predictions of the effective electroelastic moduli of piezoelectric composites. *Int. J. Solids Structures* **30,** 161–175.

Dunn, M. L. and Taya, M. (1993b). An analysis of piezoelectric composite materials containing ellipsoidal inhomogeneities. *Proc. R. Soc. London* **443A,** 265–287.

Dvorak, G. J. and Benveniste, Y. (1992). On transformation strains and uniform fields in multiphase elastic media. *Proc. R. Soc. London* **437A,** 291–310.
Getman, I. P. and Mol'kov, V. A. (1992). The effective characteristics of piezoactive composites with cylindrical inclusions. *J. Appl. Math. Mech.* **56,** 415–423.
Grekov, A. A., Kramarov, S. O. and Kuprienko, A A. (1989). Effective properties of a transversely isotropic piezoelectric composite with cylindrical inclusions. *Mech. Comp. Mater.* **25,** 54–60.
Gururaja, T. R., Schulze, W. A., Shrout, T. R., Safari, A., Webster, L. and Cross, L. E. (1981). High frequency applications of PZT/polymer composite materials. *Ferroelectrics* **39,** 1245–1248.
Hill, R. (1964). Theory of mechanical properties of fibre-strengthened materials. I. Elastic behaviour. *J. Mech. Phys. Solids* **12,** 199–212.
Hill, R. (1965a). Theory of mechanical properties of the fibre-strengthened materials. III. Self-consistent model. *J. Mech. Phys. Solids* **13,** 189–198.
Hill, R. (1965b). A self-consistent mechanics of composite materials. *J. Mech. Phys. Solids* **13,** 213–222.
Laws, N. (1974). The overall thermoelastic moduli of transversely isotropic composites according to the self-consistent method. *Int. J. Engng Sci.* **12,** 79–87.
Levin, V. M. (1967). Thermal expansion coefficients of heterogeneous materials. *Mekhanika Tverdogo Tela* **2,** 88–94.
Nan, C.-E. (1993). Comment on "Relationships between the effective properties of transversely isotropic piezoelectric composites". *J. Mech. Phys. Solids* **41,** 1567–1570.
Rosen, B. W. and Hashin, Z. (1970). Effective thermal expansion coefficients and specific heats of composite materials. *Int. J. Engng Sci.* **8,** 157–173.
Schulgasser, K. (1992). Relationships between the effective properties of transversely isotropic piezoelectric composites. *J. Mech. Phys. Solids* **40,** 473–479.
Smith, W. A. (1989). The role of piezocomposites in ultrasonic transducers. IEEE Ultrasonics Symp., pp. 755–766, New York.
Tiersten, H. F. (1969). *Linear Piezoelectric Plate Vibrations.* Plenum Press, New York.
Wang, B. (1992). Three-dimensional analysis of an ellipsoidal inclusion in a piezoelectric material. *Int. J. Solids Structures* **29,** 293–308.